THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MMAT5540 Advanced Geometry 2016-2017
Suggested Solution to Assignment 4

1. Let $x(t) = \frac{t}{2}$ and $y(t) = \frac{1}{2}$. Then, $\frac{dx}{dt} = \frac{1}{2}$ and $\frac{dy}{dt} = 0$. Then,

$$\begin{aligned}
\text{Length of } \gamma &= \int_\gamma ds \\
&= \int_0^1 \frac{2}{1-[x(t)]^2-[y(t)]^2}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt \\
&= \int_0^1 \frac{2}{1-\frac{t^2}{4}-\frac{1}{4}}\cdot\frac{1}{2}\,dt \\
&= \int_0^1 \frac{4}{3-t^2}\,dt \\
&= \frac{2}{\sqrt{3}}\int_0^1 \frac{1}{\sqrt{3}+t}+\frac{1}{\sqrt{3}-t}\,dt \\
&= \frac{2}{\sqrt{3}}\Big[\ln|\sqrt{3}+t|-\ln|\sqrt{3}-t|\Big]_0^1 \\
&= \frac{2}{\sqrt{3}}\ln\frac{\sqrt{3}+1}{\sqrt{3}-1}
\end{aligned}$$

2. Let $z = Re^{i\alpha} = R(\cos\alpha + i\sin\alpha)$ for some $\alpha \in \mathbb{R}$, $0 < R < 1$. Then,

$$\begin{aligned}
d(0,z) = \text{Length of } \gamma &= \int_\gamma ds \\
&= \int_0^R \frac{2}{1-r^2}\,dr \\
&= \int_0^R \frac{1}{1+r}+\frac{1}{1-r}\,dt \\
&= [\ln|1+r|-\ln|1-r|]_0^R \\
&= \ln\frac{1+R}{1-R}
\end{aligned}$$

3. (a) The P-line passing through $z_1 = \frac{i}{2}$ and $z_2 = \frac{1}{2}+\frac{i}{2}$ is the intersection of the circle passing through z_1, z_2 and $\frac{1}{\overline{z_1}} = 2i$ and $\mathbb{D}$. The required equation is

$$(x-\frac{1}{4})^2+(y-\frac{5}{4})^2 = \frac{5}{8}.$$

(b) Let $f(z) = \frac{z-z_1}{\overline{z_1}z-1}$. Then $f(z_1) = 0$ and $f(z_2) = \frac{z_2-z_1}{\overline{z_1}z_2-1}$. We have,

$$\begin{aligned}
d(z_1,z_2) &= d(0,f(z_2)) \\
&= \ln\frac{1+|f(z_2)|}{1-|f(z_2)|} \\
&= \ln\frac{5+\sqrt{10}}{5-\sqrt{10}}
\end{aligned}$$

(Remark: The answer obtained is less than the one obtained in question 1.)

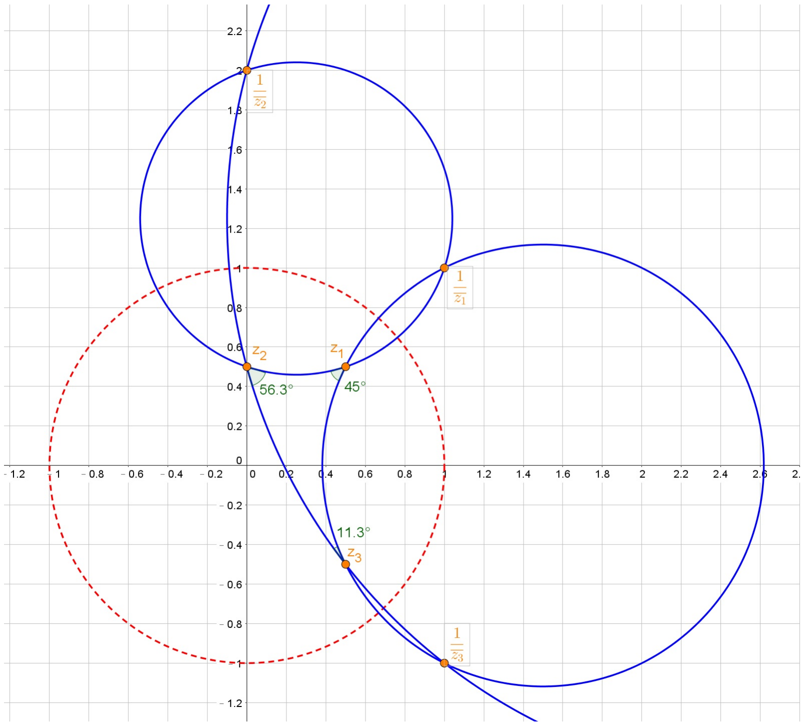
4. (a) Let $f(z) = \dfrac{z - z_1}{\overline{z_1}z - 1} = \dfrac{2z - (1+i)}{(1-i)z - 2}$.

   Then, we have $A' = f(z_1) = 0$, $B' = f(z_2) = \dfrac{3}{5} + \dfrac{1}{5}i \approx 0.632(\cos 18.4^\circ + i \sin 18.4^\circ)$ and $C' = f(z_3) = \dfrac{2}{5} + \dfrac{4}{5}i \approx 0.894(\cos 63.4^\circ + i \sin 63.4^\circ)$.

   Therefore, P-angle $\angle BAC = P$-angle $\angle B'A'C' \approx 45.0^\circ$.

   (b) By using the similar method, we have P-angle $\angle ABC \approx 56.3^\circ$ and P-angle $\angle ACB \approx 11.3^\circ$.

   Therefore, the sum of interior P-angles of P-triangle $\triangle ABC$ is 112.6°.



5. Let γ be the P-circle centered at C with P-line segment CA as radius.

   Let $f(z) = \dfrac{z - \frac{i}{2}}{(-\frac{i}{2})z - 1} = \dfrac{2z - i}{-iz - 2}$. Then $f(\dfrac{1}{2} - \dfrac{i}{2}) = -\dfrac{3}{13} + \dfrac{11}{13}i$. The image of γ under $f(z)$ is a P-circle given by the equation

$$x^2 + y^2 = \left| f(\frac{1}{2} - \frac{i}{2}) \right|^2 = \frac{10}{13}.$$

   Let $z = x + iy = f(w) = f(u + iv)$, where $x, y, u, v \in \mathbb{R}$. Then,

$$\begin{aligned} x + iy &= \frac{2(u+iv) - i}{-i(u+iv) - 2} \\ &= \frac{2u + (2v-1)i}{(v-2) - iu} \\ &= \frac{-3u + (2u^2 + 2v^2 - 5v + 2)i}{u^2 + (v-2)^2} \end{aligned}$$

Therefore, $x = \dfrac{-3u}{u^2+(v-2)^2}$ and $y = \dfrac{2u^2+2v^2-5v+2}{u^2+(v-2)^2}$. Then,

$$\begin{aligned}
x^2+y^2 &= \frac{10}{13} \\
\left[\frac{-3u}{u^2+(v-2)^2}\right]^2 + \left[\frac{2u^2+2v^2-5v+2}{u^2+(v-2)^2}\right]^2 &= \frac{10}{13} \\
13\left[(-3u)^2+(2u^2+2v^2-5v+2)^2\right] &= 10\left[u^2+(v-2)^2\right]^2 \\
13\left[u^2+(v-2)^2\right](4u^2+4v^2-4v+1) &= 10\left[u^2+(v-2)^2\right]^2 \\
\left[u^2+(v-2)^2\right]\left[13(4u^2+4v^2-4v+1)-10(u^2+(v-2)^2)\right] &= 0 \\
42\left[u^2+(v-2)^2\right]\left[u^2+v^2-\frac{2}{7}v-\frac{9}{14}\right] &= 0 \\
42\left[u^2+(v-2)^2\right]\left[u^2+(v-\frac{1}{7})^2-\frac{65}{98}\right] &= 0 \\
u^2+(v-\frac{1}{7})^2 &= \frac{65}{98}
\end{aligned}$$

The equation of γ is given by $u^2+(v-\frac{1}{7})^2 = \frac{65}{98}$.

6. Let a be the center of Γ. Note that $z_1 = \frac{3}{4}$ and $z_2 = \frac{1}{4}$ are points lying on Γ and the P-line segment joining them is a diameter. Also, the P-line segment joining z_1 and z_2 are exactly the ordinary line segment joining them since 0, z_1 and z_2 are collinear. Therefore, $a \in \mathbb{R}$ and $\frac{1}{4} < a < \frac{3}{4}$.

   The diameter of Γ is $d(0,z_1) - d(0,z_2) = \ln\frac{1+3/4}{1-3/4} - \ln\frac{1+1/4}{1-1/4} = \ln\frac{21}{5}$.

   Therefore, the radius of the P-circle is $\frac{1}{2}\ln\frac{21}{5}$ and the distance bewteen the center of Γ and 0 is

$$d(0,a) = d(0,z_2) + d(z_2,a) = \ln\frac{5}{3} + \frac{1}{2}\ln\frac{21}{5} = \ln\frac{\sqrt{105}}{3}.$$

   On the other hand, we have $d(0,a) = \ln\frac{1+a}{1-a}$. As a result,

$$\begin{aligned}
\ln\frac{1+a}{1-a} &= \ln\frac{\sqrt{105}}{3} \\
\frac{1+a}{1-a} &= \frac{\sqrt{105}}{3} \\
a &= \frac{\sqrt{105}-3}{\sqrt{105}+3} \\
&\approx 0.547
\end{aligned}$$

   (Remark: The center of the P-circle Γ does not coincide with the center $\frac{1}{2}$ if Γ is regarded as an ordinary circle on $\mathbb{C}$.)